

# $M$ -IDEALS IN NON-UNITAL ORDERED NORMED SPACES

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**ABSTRACT.** In this paper, we study some order theoretic properties of  $M$ -ideals in order smooth  $\infty$ -normed spaces. We obtain an order theoretic version of the Alfsen-Effros' cone decomposition theorem [3, Theorem 2.9] for order smooth 1-normed spaces satisfying condition (OS.1.2). As an application of this result, we sharpen a result on the extension of bounded positive linear functionals on subspaces of order smooth  $\infty$ -normed spaces. We also give two different characterizations for  $M$ -ideals of order smooth  $\infty$ -normed spaces. Finally, we characterize approximate order unit spaces as those order smooth  $\infty$ -normed spaces  $V$  that are  $M$ -ideals in  $\tilde{V}$ . Here  $\tilde{V}$  is the order unit space obtained by adjoining an order unit to  $V$ . We obtain this result by realising a complete order smooth  $\infty$ -normed space  $V$  as  $A_0(Q(V))$ , the space of continuous affine functions on  $Q(V)$  vanishing at 0. Here  $Q(V)$  is the set of quasi-states of  $V$ .

## 1. INTRODUCTION

In 1951, R. V. Kadison proved that the self adjoint part of a unital  $C^*$ -algebra forms an Archimedean order unit [11]. This observation attracted several mathematicians' interest in the study convexity theory and ordered normed spaces. This resulted in a vast theory. For details, we refer to [7, 2, 5, 8, 6, 16, 4] to cite a few.

A closed subspace  $N$  of a real Banach space  $X$  is said to be an  $L$ -summand if there exists a closed subspace  $N'$  so that  $X = N \oplus_1 N'$ . A closed subspace  $J$  of a real Banach space  $X$  is said to be an  $M$ -ideal if  $J^\perp$  (Annihilator of  $J$ ) is an  $L$ -summand in  $X^*$ . The notion of  $M$ -ideals in a Banach space was introduced by E. M. Alfsen and E. G. Effros in their seminal paper [3] in 1972. They defined the concept solely in terms the norm of the Banach space, deliberately avoiding any extra structure. Over the years the  $M$ -ideals have been extensively studied, resulting in a vast theory. This is an important tool in Functional Analysis. For a comprehensive treatment and for references to the extensive literature on the subject one may refer to the book by P. Harmand, D. Werner and W. Werner [9] and suitable references therein.

In 2010, the second author defined order smooth  $p$ -normed spaces ( $1 \leq p \leq \infty$ ) so that every order unit space is an order smooth  $\infty$ -normed space and that every base normed space is an order smooth 1-normed space. He studied the duality between order smooth  $p$ -normed space and order smooth  $p'$ -normed space, where  $\frac{1}{p} + \frac{1}{p'} = 1$  (c. f. [14]). This generalizes the duality between order unit space and base normed space established by Ellis [7] in 1964 and thus proposing a non-unital ( $p$ -theory of) ordered normed spaces. In this paper we shall study  $M$  ideals in order smooth  $\infty$ -normed spaces.

The plan of the paper is as follows. In Section 2, we obtain an order theoretic version of the Alfsen-Effros's cone decomposition theorem [3, Theorem 2.9] for order smooth 1-normed spaces satisfying condition (OS.1.2). As an application of this result, we sharpen a result on the extension of bounded positive linear functionals on subspaces of order smooth  $\infty$ -normed spaces [14, Theorem 4.3]. As another application, we show that a

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semi  $L$ -summand of an order smooth 1-normed space  $V$  satisfying condition (OS.1.2) is an order smooth order ideal of  $V$  satisfying (OS.1.2).

In Section 3, we give two characterizations for  $M$ -ideals of order smooth  $\infty$ -normed spaces. The first, in terms of split faces and the second in terms of affinity of certain restriction of positive elements. Both the results are generalizations of their counterparts for order unit spaces proved by Alfsen and Effros [3]. These results explore some intrinsic properties enjoyed by order unit spaces.

In Section 4, we characterize approximate order unit spaces as those order smooth  $\infty$ -normed spaces  $V$  that are  $M$ -ideals in  $\tilde{V}$ . Here  $\tilde{V}$  is the order unit space obtained by adjoining an order unit to  $V$ . This result we obtain by realising complete order smooth  $\infty$ -normed spaces as certain function spaces.

## 2. CONE-DECOMPOSITION PROPERTY

Let  $X$  be a normed linear space. A non empty subset  $C \subseteq X$  is a *cone* if  $\lambda C \subseteq C$  for all  $\lambda \geq 0$  and  $C + C \subseteq C$ . A cone is called *proper* if  $C \cap -C = \{0\}$ . A convex subset  $F$  of a convex set  $S$  is called a *face* if for any  $x, y \in S$  implies  $x, y \in F$  whenever  $\lambda x + (1 - \lambda)y \in F$  for some  $0 < \lambda < 1$ . If  $S$  is a convex subset of  $X$ , then  $\text{cone}(S) := \cup_{\lambda \geq 0} \lambda S$  is the smallest cone containing  $S$ . Let  $X_1$  denote the closed unit ball of  $X$ . We say that a cone  $C$  in  $X$  is *facial* if  $C = \{0\}$  or  $C = \text{cone}(F)$  for some proper face  $F$  of  $X_1$ . Any facial cone is a proper. If  $x \in X$  and  $x \neq 0$ , then  $\text{face}_{X_1}(\frac{x}{\|x\|})$  denotes the smallest face of  $X_1$  containing  $\frac{x}{\|x\|}$ . We have  $y \in \text{face}_{X_1}(\frac{x}{\|x\|})$  if and only if  $\frac{x}{\|x\|} = \lambda y + (1 - \lambda)z$  for some  $\lambda \in (0, 1)$  and some  $z \in X_1$ . We write  $C(x) := \text{cone}(\text{face}_{X_1}(\frac{x}{\|x\|}))$  for the smallest facial cone containing  $x$ . These notions and facts can be found with details in [3]. The following result will be used frequently in this paper.

**Lemma 2.1.** [3, Lemma 2.3] *Suppose that  $p_1, \dots, p_n \in X$ . Then the following are equivalent:*

- (1)  $p_1, \dots, p_n \in C(p_1 + \dots + p_n)$ .
- (2)  $\|\sum_{i=1}^n p_i\| = \sum_{i=1}^n \|p_i\|$ .

The complementary set  $C'$  of a given cone  $C$ , is defined to be the set

$$C' = \{x \in X : C \cap C(x) = \{0\}\}.$$

**Theorem 2.2.** [3, p-107, Theorem 2.9] *Suppose that  $C$  is a norm-closed convex cone in a Banach space  $X$ . Then every  $p \in X$  admits a decomposition  $p = q + r$ ,  $\|p\| = \|q\| + \|r\|$ , where  $q \in C$  and  $r \in C'$ .*

For details one can see [3].

We now recall some definitions and facts discussed in [14].

**Definition 2.3.** [14] *Let  $(V, V^+)$  be a real ordered vector space such that  $V^+$  is a proper generating and let  $\|\cdot\|$  be a norm on  $V$  such that  $V^+$  is closed. For fixed real number  $p, 1 \leq p < \infty$ , consider the following conditions on  $V$ :*

- (1) (O.p.1) *For  $u, v, w$  with  $u \leq v \leq w$ , we have  $\|v\| \leq (\|u\|^p + \|w\|^p)^{\frac{1}{p}}$ .*
- (2) (O.p.2) *For  $v \in V$  and  $\epsilon > 0$ , there are  $v_1, v_2 \in V^+$  such that  $v = v_1 - v_2$  and  $(\|v_1\|^p + \|v_2\|^p)^{\frac{1}{p}} \leq \|v\| + \epsilon$ .*
- (3) (OS.p.2) *For  $v \in V$ , there are  $v_1, v_2 \in V^+$  such that  $v = v_1 - v_2$  and  $\|v\| = (\|v_1\|^p + \|v_2\|^p)^{\frac{1}{p}}$ .*

For  $p = \infty$ , consider the similar conditions on  $V$ :

- (1) (O. $\infty$ .1) *For  $u, v, w$  with  $u \leq v \leq w$ , we have  $\|v\| \leq \max\{\|u\|, \|w\|\}$ ,*

- (2) (O.∞.2) For  $v \in V$  and  $\epsilon > 0$ , there exist  $v_1, v_2 \in V^+$  such that  $v = v_1 - v_2$  and  $\max\{\|v_1\|, \|v_2\|\} \leq \|v\| + \epsilon$ .
- (3) (OS.∞.2) For  $v \in V$ , there are  $v_1, v_2 \in V^+$  such that  $v = v_1 - v_2$  and  $\|v\| = \max(\|v_1\|, \|v_2\|)$ .

**Theorem 2.4.** [14] Let  $(V, V^+)$  be a real ordered vector space such that  $V^+$  is proper and generating. Let  $\|\cdot\|$  be a norm on  $V$  such that  $V^+$  is closed. For each  $p, 1 \leq p \leq \infty$ , we have

- (1)  $\|\cdot\|$  satisfies (O.p.1) condition on  $V$  if and only if  $\|\cdot\|^*$  satisfies the condition (OS.p'.2) on the Banach dual  $V^*$ .
- (2)  $\|\cdot\|$  satisfies the condition (O.p.2) on  $V$  if and only if  $\|\cdot\|^*$  satisfies the condition (O.p'.1) on  $V^*$ .

**Definition 2.5.** [14] Let  $(V, V^+)$  be real ordered vector space such that  $V^+$  is proper generating and let  $\|\cdot\|$  be norm on  $V$  such that  $V^+$  is closed. For a fixed  $p, 1 \leq p \leq \infty$ , we say that  $V$  is an order smooth  $p$ -normed space, if  $\|\cdot\|$  satisfies the conditions (O.p.1) and (O.p.2) on  $V$ .

**Theorem 2.6.** [14] Let  $(V, V^+)$  be a real ordered vector space such that  $V^+$  is proper and generating and let  $\|\cdot\|$  be norm on  $V$  such that  $V^+$  is closed. For a fixed  $p, 1 \leq p \leq \infty$ ,  $V$  is an order smooth  $p$ -normed space if and only if its Banach dual  $V^*$  is an order smooth  $p'$ -normed space satisfying the condition (OS.p'.2).

Let  $(V, V^+)$  be a real ordered vector space then an element  $e \in V^+$  is called order unit if for each  $v \in V$ , there exist positive real number  $r$  such that  $-re \leq v \leq re$ . Let  $(V, V^+)$  be a real vector space, with an order unit  $e$ . Then  $e$  is called an Archimedean order unit whenever  $v \in V$  with  $re + v \geq 0$  for all  $r > 0$ , implies  $v \in V^+$ . Every order unit element of a order vector space gives semi norm  $\|\cdot\|_e$ . And we know that  $e$  is Archimedean order unit iff  $\|\cdot\|_e$  is norm. Now we shall describe another class of order vector spaces. Let  $(V, V^+)$  be a real ordered vector space. A nonempty convex subset  $B$  of  $V^+$  is called a base for  $V^+$  if for every  $x \in V^+, x \neq 0$  has a unique representation  $x = \lambda x_0$ , where  $x_0 \in B$  and  $\lambda$  is positive. For more details we refer the reader to [1, 7, 10, 17].

**Theorem 2.7.** [7] Let  $(V, e)$  be order unit space, then  $(V^*, K)$  is base normed space and base normed is the usual norm of  $V^*$  considered as a dual of  $V$  with order unit norm. Conversely, if  $(V, S)$  is base normed space then  $(V^*, e_S)$  is the order unit space and order unit norm is the usual norm of  $V^*$  considered as the dual norm of  $V$  with base norm.

Let  $V$  be a (real) ordered space. An increasing net  $\{e_\lambda\} \in V^+$  is called an approximate order unit for  $V$  if for each  $v \in V$  there is  $k > 0$  such that  $ke_\lambda \pm v \in V^+$  for some  $\lambda$ . In this case  $\{e_\lambda\}$  determines a semi norm  $\|\cdot\|_a$  on  $V$  that satisfies (O.∞.1) and (O.∞.2). We call  $(V, \{e_\lambda\})$  an approximate order unit space if  $\|\cdot\|_a$  is a norm on  $V$  in which  $V^+$  is closed. It may be noted that the self-adjoint part of a  $C^*$ -algebra is an approximate order unit space. We know that when  $V$  is an approximate order unit space, its dual  $V^*$  is base normed space [7, 17].

Now we shall prove an order theoretic version of the ‘Alfsen-Effros’ cone decomposition theorem 2.2 for order smooth 1-normed spaces which satisfies (OS.1.2).

**Theorem 2.8.** Let  $(V, V^+)$  complete order smooth 1-normed space satisfying (OS.1.2) and  $W$  be a closed cone of  $V$ . Then for any  $v \in V^+$ , there are  $w \in W^+, w' \in W'^+$  such that  $v = w + w'$  and  $\|v\| = \|w\| + \|w'\|$ .

We need the following result.

**Lemma 2.9.** *Let  $(V, V^+)$  be order smooth 1-normed space satisfying (OS.1.2). If  $x \geq 0$  then  $\text{face}_{V_1}(\frac{x}{\|x\|}) \subseteq V^+$ .*

*Proof.* Let  $x \in V^+$ . Without any loss of generality, we may assume that  $\|x\| = 1$ . Let  $y \in \text{face}_{V_1}(x)$ . Then by the definition of  $\text{face}_{V_1}(x)$ , there exists  $z \in V_1$  such that

$$(1) \quad x = \lambda y + (1 - \lambda)z \text{ for some } \lambda \in (0, 1).$$

Since  $\|x\| = 1$ , we have  $\|y\| = 1 = \|z\|$ . Also, as  $V$  satisfies (OS.1.2), there exist  $y_1, y_2, z_1, z_2 \in V^+$  such that

$$\begin{aligned} y &= y_1 - y_2 & \|y\| &= \|y_1\| + \|y_2\| \\ z &= z_1 - z_2 & \|z\| &= \|z_1\| + \|z_2\| \end{aligned}$$

Thus  $x = x_1 - x_2$  where  $x_i = \lambda y_i + (1 - \lambda)z_i$  for  $i = 1, 2$ . Since  $0 \leq x \leq x_1$  and since  $V$  is an order smooth 1-normed space, we have

$$\begin{aligned} 1 &= \|x\| \leq \|x_1\| \\ &\leq \|\lambda y_1 + (1 - \lambda)z_1\| \\ &\leq \lambda\|y_1\| + (1 - \lambda)\|z_1\| \\ &\leq \lambda(\|y_1\| + \|y_2\|) + (1 - \lambda)(\|z_1\| + \|z_2\|) \\ &\leq \lambda\|y\| + (1 - \lambda)\|z\| = 1. \end{aligned}$$

Thus  $y_2 = 0 = z_2$  so that  $y, z \in V^+$ . □

*Proof of Theorem 2.8.* Let  $W$  be a closed cone of  $V$  and  $x \in V^+$ . Then by theorem 2.2, we have  $x = p + q$  with  $\|x\| = \|p\| + \|q\|$ , for some  $p \in W$  and  $q \in W'$ . Now, by Lemmas 2.1 and 2.9, we can conclude that  $p$  and  $q \in V^+$ . □

A quick consequence of Lemma 2.9 is the following:

**Corollary 2.10.** *Let  $(V, V^+)$  be complete order smooth 1-normed space satisfying (OS.1.2). Then  $\text{face}_{V_1}(\frac{x}{\|x\|}) = \text{face}_{Q(V)}(\frac{x}{\|x\|})$  and  $C(x) \in V^+$  whenever  $x \in V^+$ . Here  $Q(V) = V_1 \cap V^+$ .*

**Corollary 2.11.** *Let  $(V, V^+)$  be complete order smooth 1-normed space satisfying (OS.1.2), then we have  $(-V^+)' = V^+, (V^+)' = -V^+$ .*

*Proof.* We shall prove only  $(-V^+)' = V^+$ , as same arguments will work for other one. Assume that  $W = -V^+$  and let  $x \in V^+$ . By Theorem 2.8, we have,  $x = y + z$  with  $\|x\| = \|y\| + \|z\|$  for some  $y \in W^+, z \in W'^+$ . But  $W^+ = W \cap V^+ = -V^+ \cap V^+ = \{0\}$  so that  $x = z \in W'^+$ . Conversely, let  $y \in W' := (-V^+)'$ . Then by the definition,  $C(y) \cap (-V^+) = \{0\}$ . Since  $V$  satisfies (OS.1.2), there are  $y_1, y_2 \in V^+$  such that  $y = y_1 - y_2$  and  $\|y\| = \|y_1\| + \|y_2\|$ . By Lemma 2.1,  $-y_2 \in C(y)$ . But  $C(y) \cap -V^+ = \{0\}$  so that  $y = y_1 \in V^+$ . □

As an application of the order cone decomposition theorem 2.8, we sharpen [14, Theorem 4.3]. Actually, we prove positive and norm preserving extensions of positive bounded linear functionals without the assumption that the order smooth subspace be “strong”.

**Theorem 2.12.** *Let  $W$  be order smooth subspace of an order smooth  $\infty$ -normed space  $(V, V^+)$ . Then every positive bounded linear functional on  $W$  has positive norm preserving extension on  $V$ .*

*Proof.* Let  $f$  be a positive bounded linear functional on  $W$ . By Hahn Banach theorem there exists  $F \in V^*$  such that  $\|F\| = \|f\|$ . We prove that  $F$  is positive. Since  $V^*$  satisfy (OS.1.2), by theorem 2.6, there are  $F_1, F_2 \in V^{*+}$  such that  $F = F_1 - F_2$  with  $\|F\| = \|F_1\| + \|F_2\|$ . Since  $F_1, F_2 \in V^+$  and  $V^*$  is complete, By the order cone decomposition theorem 2.8, there are  $F_{11}, F_{21} \in W^{\perp+}$  and  $F_{12}, F_{22} \in W^{\perp'+}$  such that

$$\begin{aligned} F_1 &= F_{11} + F_{12} & \|F_1\| &= \|F_{11}\| + \|F_{12}\| \\ F_2 &= F_{21} + F_{22} & \|F_2\| &= \|F_{21}\| + \|F_{22}\|. \end{aligned}$$

Now  $F = F_{11} - F_{21} + F_{12} - F_{22}$ , where  $F_{11}, F_{21} \in W^{\perp+}$  and  $F_{12}, F_{22} \in W^{\perp'+}$  such that  $\|F\| = \|F_{11}\| + \|F_{21}\| + \|F_{12}\| + \|F_{22}\|$ . If  $f_{ij} = F_{ij}|_W$  for all  $i, j \in \{1, 2\}$ . Then  $f_{11} = f_{21} = 0$ , so that  $f = f_{12} - f_{22}$ . Further, as  $f$  is positive, we get  $\|f\| \leq \|f_{12}\|$ . Thus we have

$$\begin{aligned} \|f\| &\leq \|f_{12}\| \\ &\leq \|F_{11}\| + \|F_{21}\| + \|F_{12}\| + \|F_{22}\| \\ &= \|F\| = \|f\| \end{aligned}$$

and consequently,  $F_{11} = F_{21} = F_{22} = 0$ . Hence  $F = F_{12} \in V^{*+}$ .  $\square$

### 3. *M-ideals* in order smooth $\infty$ -normed space

Let us recall the following characterization of *M-ideals* of a complete approximate order unit space.

**Definition 3.1.** Let  $V$  be a normed linear space and let  $K$  be a non-empty, closed, convex set in  $V$ . A proper face  $F$  of  $K$  is said to be a split face of  $K$  if  $F^C$  is a proper face of  $K$  such that  $K = F \oplus_c F^C$ . Here

$$F^C = \cup \{ \text{face}_K(v) : v \in K \text{ and } \text{face}_K(v) \cap F = \emptyset \}$$

and by  $K = F \oplus_c F^C$ , we mean that for each  $k \in K$  there exist unique  $v \in F, w \in F^C$  and  $\lambda \in [0, 1]$  such that  $k = \lambda v + (1 - \lambda)w$ .

**Theorem 3.2.** [3, Corollary 5.9] Let  $W$  be a closed subspace of a complete approximate order unit space  $(V, \{e_\lambda\})$ . Then  $W$  is an *M-ideal* in  $V$  if and only if  $W^\perp \cap K$  is closed split face of  $K$ , Where  $K$  is the state space of  $V$ .

In the first part of this section, we shall prove an analog of this result for complete order smooth  $\infty$ -normed spaces. Let us begin with the following definition.

**Definition 3.3.** Let  $V$  be order smooth  $p$ -normed space and  $C$  and  $D$  be subsets of  $V^+$ . We shall write  $V^+ = C \oplus_{c,p} D$ , if every element  $u$  of  $V^+$  can be written uniquely as  $u = v + w$  with  $\|u\|^p = \|v\|^p + \|w\|^p$ , where  $v \in C$  and  $w \in D$ .

**Theorem 3.4.** Let  $V$  be a complete order smooth  $\infty$ -normed space and  $W$  be a closed subspace of  $V$ . Then  $W$  is an *M-ideal* of  $V$  if and only if  $W$  satisfies the following conditions.

- (1)  $W^{\perp'+}$  is convex.
- (2)  $V^{*+} = W^{\perp+} \oplus_{c,1} W^{\perp'+}$ .
- (3) If  $f \in W^{\perp+}$  and  $g \in W^{\perp'+}$ , then  $\|f + g\| = \|f\| + \|g\|$ .
- (4)  $W^\perp$  satisfies (OS.1.2).

*Proof.* Let  $W$  be an  $M$ -ideal of  $V$ . Then  $W^\perp$  is an  $L$ -summand of  $V^*$  so that  $W^{\perp'}$  is also an  $L$ -summand of  $V^*$  with  $V^* = W^\perp \oplus_1 W^{\perp'}$ . Thus  $W^{\perp'+} = W^{\perp'} \cap V^{*+}$  is convex. Also, by the order cone decomposition theorem 2.2, conditions (2) and (3) hold. Further, as every  $L$ -summand is a semi  $L$ -summand, condition (4) will hold, by Proposition 3.9.

Conversely, assume that conditions (1)–(4) hold. Let  $f \in V'^+$ . Then by condition (2), there exist unique  $g \in W^{\perp+}$ , and  $h \in W^{\perp'+}$  such that  $f = g + h$  with  $\|f\| = \|g\| + \|h\|$ . Let us write  $L_0(f) = g$ . Then by the uniqueness of decomposition  $L_0 : V^+ \mapsto V^+$  is well defined and  $L_0(\alpha f) = \alpha L_0(f)$  for all  $\alpha \geq 0$ . Now, let  $f_1, f_2 \in V^+$ . Again applying condition (2), we can find unique  $g_1, g_2 \in W^{\perp+}$  and  $h_1, h_2 \in W^{\perp'+}$  such that  $f_i = g_i + h_i$  and  $\|f_i\| = \|g_i\| + \|h_i\|$  for  $i = 1, 2$ . Then  $f_1 + f_2 = (g_1 + g_2) + (h_1 + h_2)$ , where  $g_1 + g_2 \in W^{\perp+}$  and  $h_1 + h_2 \in W^{\perp'+}$  by the condition (1). Thus by the condition (3), we have  $\|f_1 + f_2\| = \|(g_1 + g_2)\| + \|(h_1 + h_2)\|$  so that  $L_0(f_1 + f_2) = L_0(f_1) + L_0(f_2)$ . Now, let  $f \in V^*$ . By the condition (OS.1.2) in  $V^*$ , there are  $f_1, f_2 \in V^{*+}$  such that  $f = f_1 - f_2$  with  $\|f\| = \|f_1\| + \|f_2\|$ . Let us write  $L(f) = L_0(f_1) - L_0(f_2)$ . As  $L_0$  is additive on  $V^{*+}$ , it is routine to check that  $L : V^* \rightarrow V^*$  is a well defined, positive linear mapping with  $L(V^*) \subset W^\perp$ . We prove that  $L$  is  $L$ -projection on  $W^\perp$ . Let  $f \in V'$ , then by (OS.1.2) in  $V^*$ , there are  $g, h \in V'^+$  such that  $f = g - h$  with  $\|f\| = \|g\| + \|h\|$ . Now

$$\begin{aligned} \|f\| &\leq \|L(f)\| + \|f - L(f)\| = \|L(g) - L(h)\| + \|g - h - L(g) + L(h)\| \\ &\leq \|L(g)\| + \|L(h)\| + \|g - L(g)\| + \|h - L(h)\| \\ &= (\|L_0(g)\| + \|g - L_0(g)\|) + (\|L_0(h)\| + \|h - L_0(h)\|) \\ &= \|g\| + \|h\| = \|f\| \end{aligned}$$

so that  $\|L(f)\| + \|f - L(f)\| = \|f\|$  for all  $f \in V^*$ . Next, we show that  $L(f) = f$  for all  $f \in W^\perp$ . To see this, let  $f \in W^\perp$ . Then by condition (4), there are  $f_1, f_2 \in W^{\perp+}$  such that  $f = f_1 - f_2$  with  $\|f\| = \|f_1\| + \|f_2\|$ . By the order cone decomposition theorem  $f_i = g_i + h_i$ , where  $g_i \in W^{\perp+}$  and  $h_i \in W^{\perp'+}$  for  $i = 1, 2$ . As  $f_i, g_i \in W^{\perp+}$ ,  $h_i \in W^\perp \cap W^{\perp'+} = \{0\}$ . Now, by the construction,  $g_1 = L_0(f_1)$  and  $g_2 = L_0(f_2)$  so that  $L(f) = L_0(f_1) - L_0(f_2) = g_1 - g_2 = f$  if  $f \in W^\perp$ . Thus for any  $f \in V^*$ , we have  $L^2(f) = L(L(f)) = L(f)$ . Hence  $L$  is an  $L$ -projection of  $V'$  on  $W^\perp$  and therefore  $W$  is an  $M$ -ideal of  $V$ .  $\square$

**Corollary 3.5.** *Let  $W$  be a closed subspace of order smooth  $\infty$ -normed space  $V$ . Then the following set of statements are equivalent.*

- (1)  $W$  is an  $M$ -ideal of  $V$ .
- (2)
  - (a)  $W^\perp \cap Q(V)$  is a face of  $Q(V)$ .
  - (b)  $W^{\perp'} \cap Q(V)$  is a face of  $Q(V)$ .
  - (c)  $Q(V) = W^\perp \cap Q(V) \oplus_c W^{\perp'} \cap Q(V)$ .
  - (d) For  $f \in W^{\perp+}, g \in W^{\perp'+}$ , we have  $\|f + g\| = \|f\| + \|g\|$ .
  - (e)  $W^\perp$  satisfies (OS.1.2).
- (3)
  - (a)  $W^{\perp'+} \cap Q(V)$  is a convex set.
  - (b)  $Q(V) = W^\perp \cap Q(V) \oplus_c W^{\perp'} \cap Q(V)$ .
  - (c) For  $f \in W^{\perp+}, g \in W^{\perp'+}$ , we have  $\|f + g\| = \|f\| + \|g\|$ .
  - (d)  $W^\perp$  satisfies (OS.1.2).

Here  $Q(V)$  is the quasi-state of  $V$ , that is, the set of bounded positive linear functionals on  $V$  whose norms are less than or equal to one.

*Proof.* Since condition (2) evidently implies condition (3), it suffices to prove that condition (1) implies condition (2) and that condition (3) implies condition (1).

(1) implies (2): Let  $f, g \in Q(V)$  such that  $h = \lambda f + (1 - \lambda)g \in W^\perp \cap Q(V)$  for some

$0 < \lambda < 1$ . Then by the cone decomposition theorem 2.2, there are  $f_1, g_1 \in W^\perp$  and  $f_2, g_2 \in W^{\perp'}$  such that

$$\begin{aligned} f &= f_1 + f_2, & \|f\| &= \|f_1\| + \|f_2\|. \\ g &= g_1 + g_2, & \|g\| &= \|g_1\| + \|g_2\|. \end{aligned}$$

Thus we have

$$(\lambda f_1 + (1 - \lambda)g_1 - h) + \lambda f_2 = -(1 - \lambda)g_2$$

and

$$(\lambda f_1 + (1 - \lambda)g_1 - h) + (1 - \lambda)g_2 = -\lambda f_2.$$

Since  $W^\perp$  is an  $M$ -ideal, it follows that

$$\|(\lambda f_1 + (1 - \lambda)g_1 - h)\| + \|\lambda f_2\| = \|(1 - \lambda)g_2\|$$

and

$$\|(\lambda f_1 + (1 - \lambda)g_1 - h)\| + \|(1 - \lambda)g_2\| = \|\lambda f_2\|.$$

Form last two relations, we can conclude that  $h = \lambda f_1 + (1 - \lambda)g_1$  and that  $f_2 = g_2 = 0$ . Therefore  $W^\perp \cap Q(V)$  is a face and by similar arguments, we can show that  $W^{\perp'} \cap Q(V)$  is a face of  $Q(V)$  as well. Now that  $Q(V) = W^\perp \cap Q(V) \oplus_c W^{\perp'} \cap Q(V)$ , follows from theorem 3.4.

(3) *implies* (1): Note that  $W^{\perp'} \cap Q(V)$  is convex if and only if  $W^{\perp'+}$  is convex and that  $Q(V) = W^\perp \cap Q(V) \oplus_c W^{\perp'} \cap Q(V)$  if and only if  $V'^+ = W^{\perp+} \oplus_{c,1} W^{\perp'+}$ . Thus  $W$  is an  $M$ -ideal of  $V$ , by Theorem 3.4.  $\square$

**Remark 3.6.** *It may be noted that Conditions (2) (a) – (c) of the Lemma 3.5 hold if and only if  $W^\perp \cap Q(V)$  is a split face of  $Q(V)$ . Thus the Lemma may be restated as follows:  $W$  is an  $M$ -ideal of  $V$  if and only if*

- (1)  $W^\perp \cap Q(V)$  is a split face of  $Q(V)$ .
- (2) For  $f \in W^{\perp+}, g \in W^{\perp'+}$ , we have  $\|f + g\| = \|f\| + \|g\|$ .
- (3)  $W^\perp$  satisfying (OS.1.2).

We now show that conditions (2) and (3) are redundant in the case of an approximate order unit spaces.

**Definition 3.7.** *A subspace  $Y$  of a normed linear space  $X$  is said to be hereditary, if  $x \prec y$  (equivalently,  $x \in C(y)$ ) with  $y \in Y$  implies  $x \in Y$ . Here  $x \prec y$ , if  $\|y\| = \|x\| + \|y - x\|$ .*

It may be noted that this relation is transitive.

**Proposition 3.8.** *Let  $V$  be a base normed space and  $W$  be a closed subspace of  $V$  such that  $W \cap V_1^+$  is a face in  $V_1^+$ , then  $W$  is hereditary.*

*Proof.* Let  $y \in W$  and  $x \in C(y)$ . We prove that  $x \in W$ . By (OS.1.2) property of  $V$ , we have  $x = x_1 - x_2$  with  $\|x\| = \|x_1\| + \|x_2\|$ , for some  $x_1, x_2 \in V^+$ . Thus  $x_1 \prec x \prec y$ . Since  $W^+$  is cone in  $V$ , by [3, Theorem 2.9] together with Theorem 2.8, we have  $y = y_1 + y_2$ , with  $\|y\| = \|y_1\| + \|y_2\|$  and  $x_1 \prec y_1$  where  $y_1 \in W^+$  and  $y_2 \in W^{\perp'}$ . As  $x_1 \prec y_1$ ,  $y_1 \in V^+$  and  $V$  is base normed space, we get  $x_1 \prec y_1$  and consequently,  $0 \leq x_1 \leq y_1$ . Further, as the norm is additive on  $V^+$ , the right hand side of the expression

$$\frac{y_1}{\|y_1\|} = \frac{\|x_1\|}{\|y_1\|} \left( \frac{x_1}{\|x_1\|} \right) + \frac{\|y_1 - x_1\|}{\|y_1\|} \left( \frac{y_1 - x_1}{\|y_1 - x_1\|} \right)$$

is a convex combination of  $\frac{x_1}{\|x_1\|}, \frac{y_1 - x_1}{\|y_1 - x_1\|}$  in  $V_1^+$ . Since  $W \cap V_1^+$  is a face of  $V_1^+$  and since  $y_1 \in W \cap V_1^+$ , we have  $x_1 \in W^+$ . By a similar argument, we can show that  $x_2 \in W^+$ . Hence  $x \in W$ .  $\square$

**Proposition 3.9.** *Let  $W$  be a hereditary subspace of a complete order smooth 1-normed space  $V$ . Then  $W$  satisfies (OS.1.2) if so does  $V$ .*

*Proof.* Let  $w \in W$ , then by (OS.1.2) property of  $V$ , there are  $u, v \in V^+$  such that  $w = u - v$  and  $\|w\| = \|u\| + \|v\|$ . Therefore  $u, -v \prec w$ , so by definition of hereditary subspaces,  $u, -v \in W$ . Thus  $u, v \in W^+$  so that  $W$  also satisfies (OS.1.2)  $\square$

Since the dual of an *approximate order unit space* is a *base normed space* in which norm is additive on its positive part. Now, by Propositions 3.8 and 3.9, we have the following characterization for  $M$ -ideals in an *approximate order unit space*.

**Corollary 3.10.** *Let  $W$  be a closed subspace of an approximate order unit space  $(V, V^+, \{e_\lambda\})$ . Then following statements are equivalent:*

- (1)  $W$  is an  $M$ -ideal of  $V$ .
- (2) (a)  $W^\perp \cap Q(V)$  is a face of  $Q(V)$ .  
 (b)  $W^{\perp'} \cap Q(V)$  is a face of  $Q(V)$ .  
 (c)  $Q(V) = (W^\perp \cap Q(V)) \oplus_c (W^{\perp'} \cap Q(V))$ .
- (3) (a)  $W^{\perp'} \cap Q(V)$  is a convex set.  
 (b)  $Q(V) = (W^\perp \cap Q(V)) \oplus_c (W^{\perp'} \cap Q(V))$ .

### 3.1. Another characterization of $M$ -ideals in an order smooth $\infty$ -normed space.

Now we describe another characterization. Let  $W$  be a closed subspace of a complete order smooth  $\infty$ -normed space and let  $Q(V)$  its quasi-state space. For a given  $v \in V^+$ , we define  $h_v : Q(V) \mapsto \mathbb{R}$  given by  $h_v = v\chi_{Q(V) \cap W^\perp}$ . For a given bounded above function  $f : Q(V) \rightarrow \mathbb{R}$ , its upper envelope is a function  $\widehat{f} : Q(V) \rightarrow \mathbb{R}$  defined as  $\widehat{f} = \inf\{a \in A(Q(V)) : f \leq a\}$ . This terminology we have taken from [3]. In the second part of this section, we shall characterize  $M$ -ideal using affinity of  $\widehat{h_v}$ . The forthcoming construction is influenced by [3].

**Lemma 3.11.** [3, Lemma 1.2] *Suppose that  $K$  is compact convex subset of a locally convex Hausdorff space  $E$ , and that  $f : K \rightarrow \mathbb{R}$  is a bounded above function. Then  $Sub_K^0 \widehat{f} = \bar{co}[Sub_K^0 f]$ .*

Here

$$Sub_K^\alpha f := \{(k, \gamma) : k \in K, \alpha \leq \gamma \leq f(k)\}.$$

**Lemma 3.12.** *Let  $W$  be a closed subspace of an order smooth  $\infty$ -normed space  $V$  and  $v \in V^+$ . Then for each  $(f, \gamma) \in Sub_{Q(V)}^0 \widehat{h_v}$ , we have  $(f, \gamma) = \alpha_1(f_1, f_1(v)) + \alpha_2(f_2, 0) + \alpha_3(f_3, 0)$ , for some  $f_1, f_2 \in Q(V) \cap W^\perp$ ,  $f_3 \in Q(V) \cap W^{\perp'}$  and  $\sum_{i=1}^3 \alpha_i = 1$  with  $\alpha_i \geq 0$ .*

*Proof.* First, we note that  $Sub_{Q(V)}^0 h_v = A \cup B$ , where  $A = \{(f, 0) : f \in Q(V)\}$  and  $B = \{(f, \gamma) : f \in Q(V) \cap W^\perp, 0 \leq \gamma \leq v(f)\}$ . In fact, for  $f \in Q(V)$ , we have  $h_v(f) = f(v)$  if  $f \in W^\perp$  and  $h_v(f) = 0$  if  $f \notin W^\perp$ . Now  $A$  and  $B$  are compact convex sets, so that  $co(A \cup B) = \bar{co}(A \cup B)$ . Thus by Lemma 3.11,  $Sub_{Q(V)}^0 \widehat{h_v} = co(A \cup B)$ .

Let  $(f, \gamma) \in Sub_{Q(V)}^0 \widehat{h_v}$ , then  $(f, \gamma) = \lambda(g_1, 0) + (1 - \lambda)(g_2, \mu)$ , where  $g_1 \in Q(V)$ ,  $g_2 \in Q(V) \cap W^\perp$ ,  $0 \leq \mu \leq v(g_2)$ . Now by order cone decomposition theorem 2.8,



$g_1 = g_{11} + g_{12}$ ,  $\|g_1\| = \|g_{11}\| + \|g_{12}\|$ , where  $g_{11} \in W^\perp \cap Q(V)$ ,  $g_{12} \in W^{0'} \cap Q(V)$ . Thus

$$\begin{aligned} g_1 &= \frac{\|g_{11}\|}{\|g_1\|} \left( \frac{\|g_1\| \|g_{11}\|}{\|g_{11}\|} \right) + \frac{\|g_{12}\|}{\|g_1\|} \left( \frac{\|g_1\| \|g_{12}\|}{\|g_{12}\|} \right) \\ &= \mu_1 h_{11} + (1 - \mu_1) h_{12}, \end{aligned}$$

where  $\mu_1 = \frac{\|g_{11}\|}{\|g_1\|} \in [0, 1]$ ,  $h_{11} = \|g_1\| \frac{g_{11}}{\|g_{11}\|} \in Q(V) \cap W^\perp$  and  $h_{12} = \|g_1\| \frac{g_{12}}{\|g_{12}\|} \in Q(V) \cap W^{\perp'}$ . Also,

$$(g_2, \mu) = \mu_2(g_2, 0) + (1 - \mu_2)(g_2, g_2(v))$$

for some suitable  $\mu_2 \in [0, 1]$ . Thus

$$\begin{aligned} (f, \gamma) &= \lambda(\mu_1 h_{11} + (1 - \mu_1) h_{12}, 0) + (1 - \lambda)\mu_2(g_2, 0) + (1 - \lambda)(1 - \mu_2)(g_2, g_2(v)) \\ &= \lambda\mu_1(h_{11}, 0) + \lambda(1 - \mu_1)(h_{12}, 0) + (1 - \lambda)\mu_2(g_2, 0) + (1 - \lambda)(1 - \mu_2)(g_2, g_2(v)) \\ &= (1 - \lambda)(1 - \mu_2)(g_2, g_2(v)) + (\lambda\mu_1 h_{11} + (1 - \lambda)\mu_2 g_2, 0) + \lambda(1 - \mu_1)(h_{12}, 0) \\ &= \alpha_1(f_1, f_1(v)) + \alpha_2(f_2, 0) + \alpha_3(f_3, 0). \end{aligned}$$

where  $f_1 = g_2$ ,  $f_2 = \frac{\lambda\mu_1 h_{11} + (1 - \lambda)\mu_2 g_2}{\lambda\mu_1 + (1 - \lambda)\mu_2}$ ,  $f_3 = h_{12}$ ;  $\alpha_1 = (1 - \lambda)(1 - \mu_2)$ ,  $\alpha_2 = \lambda\mu_1 + (1 - \lambda)\mu_2$  and  $\alpha_3 = \lambda(1 - \mu_1)$  so that  $f_1, f_2 \in Q(V) \cap W^\perp$ ,  $f_3 \in Q(V) \cap W^{\perp'}$  and  $\alpha_i \geq 0$  for  $i = 1, 2, 3$  with  $\sum_{i=1}^3 \alpha_i = 1$ .  $\square$

**Lemma 3.13.** *Let  $W$  be a closed subspace of an order smooth  $\infty$ -normed space  $V$  and  $v \in V^+$ . For each  $f \in Q(V)$ , there exist  $g \in Q(V) \cap W^\perp$  and  $h \in Q(V) \cap W^{\perp'}$  with  $f = g + h$  and  $\|g\| + \|h\| \leq 1$  such that  $\widehat{h}_v(f) = g(v)$ .*

*Proof.* Let  $f \in Q(V)$ . By Lemma 3.12, we have

$$(f, \widehat{h}_v(f)) = \alpha_1(f_1, f_1(v)) + \alpha_2(f_2, 0) + \alpha_3(f_3, 0),$$

where  $f_1, f_2 \in Q(V) \cap W^\perp$ ,  $f_3 \in Q(V) \cap W^{\perp'}$ ,  $\sum_{i=1}^3 \alpha_i = 1$  with  $\alpha_i \geq 0$ . Thus  $f = f_1 + f_2 + f_3$  and  $\widehat{h}_v(f) = \alpha_1 f_1(v)$ . Let  $g = \alpha_1 f_1 + \alpha_2 f_2$  and  $h = \alpha_3 f_3$ . Then  $g \in Q(V) \cap W^\perp$ ,  $h \in Q(V) \cap W^{\perp'}$  and  $\|g\| + \|h\| \leq 1$ . Now, by the definition,  $\widehat{h}_v$  is concave so that

$$\widehat{h}_v(f) \geq \sum_{i=1}^3 \alpha_i \widehat{h}_v(f_i) \geq \sum_{i=1}^2 \alpha_i \widehat{h}_v(f_i) = \alpha_1 f_1(v) + \alpha_2 f_2(v) \geq \alpha_1 f_1(v) \geq \widehat{h}_v(f).$$

Therefore  $\alpha_2 f_2(v) = 0 (= \alpha_3 f_3(v))$  and we have,  $\widehat{h}_v(f) = g(v)$ .  $\square$

**Lemma 3.14.** *Let  $W$  be a closed subspace of an order smooth  $\infty$ -normed space  $V$  and  $v \in V^+$ . Then*

- (1)  $\widehat{h}_v \Big|_{Q(V) \cap W^\perp} = v$ .
- (2)  $\widehat{h}_v \Big|_{S(V) \cap W^{\perp'}} = 0$ .

*Proof.* (1): Let  $f \in Q(V) \cap W^\perp$ . Then by Lemma 3.12, we have

$$(f, \widehat{h}_v(f)) = \alpha_1(f_1, f_1(v)) + \alpha_2(f_2, 0) + \alpha_3(f_3, 0)$$

for some  $f_1, f_2 \in Q(V) \cap W^\perp$ ,  $f_3 \in Q(V) \cap W^{\perp'}$  and  $\alpha_1, \alpha_2, \alpha_3 \geq 0$  with  $\sum_{i=1}^3 \alpha_i = 1$ . Then  $\widehat{h}_v(f) = \alpha_1 f_1(v)$  and  $f = f_1 + f_2 + f_3$  so that  $f(v) = f_1(v) + f_2(v) + f_3(v)$ . Since  $f \in Q(V) \cap W^\perp$ , we have  $\widehat{h}_v(f) \geq h_v(f) = f(v)$ . Also, as  $v \in V^+$ , we get  $f(v) \geq f_1(v)$ . Thus  $f_2(v) = 0 = f_3(v)$  and we have  $\widehat{h}_v(f) = f(v)$ .

(2): Let  $f \in W^{\perp'} \cap S(V)$ . By Lemma 3.13, we have  $\widehat{h}_v(f) = v(g)$ , where  $g \in W^\perp \cap Q(V)$ ,  $h \in W^{\perp'} \cap Q(V)$  and  $f = g + h$  with  $\|g\| + \|h\| \leq 1$ . Since  $\|f\| = 1$ , we have  $1 =$

$\|f\| = \|g\| + \|h\|$ . Thus by Lemma 2.1, we have  $g, h \in C(f)$ . As  $f \in W^{\perp'}$ , we have  $C(f) \cap W^{\perp} = \{0\}$  so that  $g = 0$ . Therefore  $\widehat{h}_v(f) = 0$ .  $\square$

**Theorem 3.15.** *Let  $W$  be a closed subspace order smooth  $\infty$ -normed space  $V$ . Then  $W$  is an  $M$ -ideal of  $V$  if and only if  $W^{\perp}$  satisfies (OS.1.2) and  $\widehat{h}_v$  is affine for each  $v \in V^+$ .*

*Proof.* First let  $\widehat{h}_v : Q(V) \mapsto \mathbb{R}$  be an affine function for all  $v \in V^+$ . Then we can extend  $\widehat{h}_v$  from  $Q(V)$  to  $V^{'+}$  by  $\widehat{H}_v(\alpha f) = \alpha \widehat{h}_v(f)$ , if  $\alpha \geq 0$  and  $f \in Q(V)$ . Since  $\widehat{h}_v$  is affine map,  $\widehat{H}_v$  is a well defined, additive and positive homogeneous map such that for all  $v \in V^+$ , we have

- (1)  $\widehat{H}_v \Big|_{W^{\perp+}} = v$ ; and
- (2)  $\widehat{H}_v \Big|_{W^{\perp'+}} = 0$ .

Let  $f \in V^+ \setminus W^{\perp'}$ . By the positive cone decomposition theorem 2.8, there exist  $g \in W^{\perp+}$ ,  $h \in W^{\perp'+}$  such that  $f = g + h$  and  $\|f\| = \|g\| + \|h\|$ . Since  $f \notin W^{\perp'+}$ , so  $g \neq 0$ . Since  $V^+$  is generating,  $g \neq 0$ . There exists  $v_0$  such that  $g(v_0) \neq 0$ . Now  $\widehat{H}_{v_0}(f) = \widehat{H}_{v_0}(g) + \widehat{H}_{v_0}(h) = g(v_0) \neq 0$ . So  $f \notin \cap_{v \in V^+} \widehat{H}_v^{-1}(0)$ . And hence  $\cap_{v \in V^+} \widehat{H}_v^{-1}(0) = W^{\perp'+}$ . Since  $\widehat{H}_v$  is an additive and positive homogeneous map,  $W^{\perp'+}$  is convex.

Let  $f \in V^{'+}$  has two decomposition  $f = g_i + h_i$  and  $\|f\| = \|g_i\| + \|h_i\|$ , where  $g_i \in W^{\perp+}$ ,  $h_i \in W^{\perp'+}$  for  $i = 1, 2$ . Then for all  $v \in V^+$ ,  $\widehat{H}_v f = \widehat{H}_v g_i + \widehat{H}_v h_i = g_i(v)$ . Therefore  $g_1(v) = g_2(v)$  for all  $v \in V^+$ . Since  $V^+$  is generating  $g_1 = g_2$  and  $h_1 = h_2$ .

Let  $f \in W^{\perp+}$ ,  $g \in W^{\perp'+}$  and let  $h = f + g$ . Since  $h \in V^+$ , then by order cone decomposition theorem 2.8, there are  $f_1 \in W^{\perp+}$ ,  $g_1 \in W^{\perp'+}$  such that  $h = f_1 + g_1$ ,  $\|h\| = \|f_1\| + \|g_1\|$ . Now  $\widehat{H}_v(h) = \widehat{H}_v(f + g) = \widehat{H}_v(f) + \widehat{H}_v(g) = f(v)$ . But  $\widehat{H}_v(h) = \widehat{H}_v(f_1 + g_1) = \widehat{H}_v(f_1) + \widehat{H}_v(g_1) = f_1(v)$ , Therefore  $f_1(v) = f(v)$ . Since  $V^+$  is generating, we get  $f = f_1$  and  $g = g_1$ . Thus  $\|f + g\| = \|f\| + \|g\|$ . Now, if  $W^{\perp}$  also satisfies (OS.1.2), then by Corollary 3.5,  $W$  is an  $M$ -ideal of  $V$ .

Conversely, let  $W$  be an  $M$ -ideal of  $V$ . Then by Corollary 3.5, we have

- (1)  $W^{\perp'} \cap Q(V)$  is a convex;
- (2)  $Q(V) = W^{\perp} \cap Q(V) \oplus_{1,c} W^{\perp'} \cap Q(V)$ ;
- (3)  $\|f + g\| = \|f\| + \|g\|$  for all  $f \in W^{\perp+}$ ,  $g \in W^{\perp'+}$ ; and
- (4)  $W^{\perp}$  satisfies (OS.1.2).

It suffices to show that  $\widehat{h}_v$  is an affine function on  $Q(V)$  for each  $v \in V^+$ . For this purpose, fix  $v \in V^+$ . Let  $f, g \in Q(V)$  and  $0 \leq \lambda \leq 1$ . Then by conditions (2) and (3), there exist unique  $f_1, g_1 \in Q(V) \cap W^{\perp}$  and  $f_2, g_2 \in Q(V) \cap W^{\perp'}$  such that  $f = f_1 + f_2$  and  $g = g_1 + g_2$  with  $\|f\| = \|f_1\| + \|f_2\|$  and  $\|g\| = \|g_1\| + \|g_2\|$ . Thus by Lemma 3.13,  $\widehat{h}_v(f) = f_1(v)$  and  $\widehat{h}_v(g) = g_1(v)$ . Now  $\lambda f + (1 - \lambda)g = (\lambda f_1 + (1 - \lambda)g_1) + (\lambda f_2 + (1 - \lambda)g_2)$ . Since  $W^{\perp'} \cap Q(V)$  is convex,  $\lambda f_2 + (1 - \lambda)g_2 \in W^{\perp'} \cap Q(V)$ . Also,  $\lambda f_1 + (1 - \lambda)g_1 \in W^{\perp} \cap Q(V)$  so that applying Lemma 3.13 again, we get

$$\begin{aligned} \widehat{h}_v(\lambda f + (1 - \lambda)g) &= (\lambda f_1 + (1 - \lambda)g_1)(v) \\ &= \lambda f_1(v) + (1 - \lambda)g_1(v) \\ &= \lambda \widehat{h}_v(f) + (1 - \lambda)\widehat{h}_v(g). \end{aligned}$$

Hence  $\widehat{h}_v : Q(V) \mapsto \mathbb{R}$  is an affine function.  $\square$

#### 4. *M-ideals* and adjoining of an order unit

Let  $V$  be a (complete) order smooth  $\infty$ -normed space and let  $\tilde{V}$  be the (complete) order unit space obtained by adjoining an order unit to  $V$ . Here  $\tilde{V} = V \oplus \mathbb{R}$ ,  $\tilde{V}^+ = \{(v, \alpha) : l_V(v) \leq \alpha\}$  and  $(0, 1)$  is the order unit element of  $\tilde{V}$  and  $l_V(v) = \inf\{\|u\| : u, u+v \in V^+\}$ . For a detailed information one can see [14, Section 4]. Throughout this section, we shall assume that all order normed spaces are complete.

Next, let  $K$  is a compact subset of a locally convex space  $E$  with  $0 \in K$ . We write

$$A_0(K) = \{f \in A(K) : f(0) = 0\}$$

and

$$A_0(K, E) = \{f \in A_0(K) : f \text{ has continuous affine real valued extension to } E\}.$$

In this section, we obtain the conditions under which  $V$  is an  $M$ -ideal in  $\tilde{V}$ . The next result (due to Alfsen and Effros) will be used for this purpose.

**Theorem 4.1.** [3, Theorem 6.10] *Let  $(V, e)$  be an order unit space and let  $W$  be a closed subspace of  $V$ . Then following sets of statements are equivalent:*

- (1)  $W$  is an  $M$ -ideal.
- (2)  $W$  satisfies each of the following conditions.
  - (a)  $W$  is positively generated.
  - (b)  $W$  is an order ideal.
  - (c)  $\pi(e)$  is an Archimedean unit for  $V/W$ .
  - (d) Given  $v, w \in V^+$  and  $\epsilon > 0$ ,

$$\pi([0, v]) \cap \pi([0, w]) \subset \pi([0, v + \epsilon e] \cap [0, w + \epsilon e]).$$

Here  $\pi : V \rightarrow V/W$  is the canonical quotient mapping.

**Proposition 4.2.** *Let  $V$  be an order smooth  $\infty$ -normed space and  $\tilde{V}$  be the order unit space obtained by adjoining an order unit to  $V$ . Then*

- (1)  $V$  is positively generated.
- (2)  $V$  is order ideal in  $\tilde{V}$ .
- (3)  $\tilde{\pi}((0, 1))$  is an Archimedean unit for  $\tilde{V}/V$ .

Here  $\tilde{\pi} : \tilde{V} \rightarrow \tilde{V}/V$  is the natural quotient mapping.

*Proof.* Condition (1) follows from the definition of  $V$  and condition (2) follows from the construction of  $\tilde{V}$ . To prove (3), first note that the natural quotient map  $\tilde{\pi} : \tilde{V} \rightarrow \tilde{V}/V$  is positive and that  $\tilde{\pi}(0, 1)$  is an order unit for  $\tilde{V}/V$ . We show that  $\tilde{\pi}(0, 1)$  is an Archimedean order unit. Let  $\tilde{\pi}(x, \alpha) \in \tilde{V}/V$  such that  $\tilde{\pi}(x, \alpha) \leq \frac{1}{n}\tilde{\pi}(0, 1)$  for all  $n \in \mathbb{N}$ . Then  $\tilde{\pi}(0, \frac{1}{n} - \alpha) = \tilde{\pi}(x, \frac{1}{n} - \alpha) \geq 0$  so that  $\frac{1}{n} - \alpha \geq 0$  for all  $n \in \mathbb{N}$ . Consequently,  $\tilde{\pi}(x, \alpha) = \tilde{\pi}(0, \alpha) \leq 0$ .  $\square$

**Lemma 4.3.** *Let  $V$  be an order smooth  $\infty$ -normed space and  $\tilde{V}$  be the order unit space obtained by adjoining an order unit to  $V$ . If  $\tilde{\pi} : \tilde{V} \rightarrow \tilde{V}/V$  is the natural quotient map, then for all  $(u, \lambda) \in \tilde{V}^+$ , we have  $\tilde{\pi}[(0, 0), (u, \lambda)] = \{\tilde{\pi}(0, \mu) : 0 \leq \mu \leq \lambda\}$ .*

*Proof.* Let us consider the order interval  $[(0, 0), (u, \lambda)]$ , where  $(u, \lambda) \in \tilde{V}^+$ . Now for any  $\mu$  in the interval  $[0, \lambda]$ , we have  $0 \leq (\frac{\mu}{\lambda}u, \mu) \leq (u, \lambda)$ . Thus  $(0, \mu) + V = \tilde{\pi}(\frac{\mu}{\lambda}u, \mu) \in \tilde{\pi}[(0, 0), (u, \lambda)]$ . Conversely, let  $(x, \mu) \in \tilde{V}$  such that  $0 \leq (x, \mu) \leq (u, \lambda)$  in  $(\tilde{V}, \tilde{V}^+)$ . Then we have  $0 \leq \mu \leq \lambda$ . Now the observation  $\tilde{\pi}(x, \mu) = \tilde{\pi}(0, \mu)$  completes the proof.  $\square$

**Proposition 4.4.** *Let  $(V, V^+, \{e_\lambda\}_{\lambda \in D})$  be an approximate order unit space, and let  $(\tilde{V}, \tilde{V}^+)$  be the order unit space obtained by adjoining an order unit to  $V$ . Then for  $(u_i, \gamma_i) \geq 0$  in  $\tilde{V}$ , for  $i = 1, 2$  we have*

$$\tilde{\pi}([0, (u_1, \gamma_1)]) \cap \tilde{\pi}([0, (u_2, \gamma_2)]) \subset \tilde{\pi}([0, (u_1, \gamma_1 + \epsilon)]) \cap [0, (u_2, \gamma_2 + \epsilon)].$$

*Proof.* Let  $l_V(u_i) \leq \gamma_i$  for  $i = 1, 2$ . Then for  $\epsilon > 0$  there exist  $e_{\lambda_i}$  for  $i = 1, 2$  such that  $-(\gamma_i + \epsilon)e_{\lambda_i} \leq u_i$ . Since  $V$  is an approximate order unit space, there exist  $e_{\lambda_3}$  such that  $-(\gamma_i + \epsilon)e_{\lambda_3} \leq u_i$  for  $i = 1, 2$ . Now consider the element  $(-\gamma e_{\lambda_3}, \gamma)$ , where  $0 \leq \gamma \leq \min\{\gamma_1, \gamma_2\}$ . Then  $l_V(-\gamma e_{\lambda_3}) \leq \gamma$  and  $u_i + \gamma e_{\lambda_3} \geq -(\gamma_i + \epsilon)e_{\lambda_i} + \gamma e_{\lambda_3} \geq -\epsilon e_{\lambda_3}$ . So we have  $(-\gamma e_{\lambda_3}, \gamma) \in \cap_{i=1}^2 [0, (u_i, \gamma_i)]$ .  $\square$

The following result is a consequence of a proposition 4.2 and proposition 4.4.

**Corollary 4.5.** *Let  $(V, V^+, \{e_\lambda\}_{\lambda \in D})$  be an approximate order unit space, and let  $(\tilde{V}, \tilde{V}^+)$  be the order unit space obtained by adjoining an order unit to  $V$ . Then  $V$  is an M-ideal in  $\tilde{V}$ .*

**Remark 4.6.** *Let  $X$  be a Banach space, and  $Y$  is closed subspace of  $X$  such that  $Y \neq 0$ . It follows from [1, Proposition 2.2] that the M-ideals of  $Y$  are precisely the M-ideals of  $X$  that are contained in  $Y$ . Thus each M-ideal of an approximate order unit space  $(V, \{e_\lambda\})$  is an M-ideal in  $(\tilde{V}, \tilde{V}^+)$ .*

**Theorem 4.7.** *Let  $(V, V^+)$  be complete ordered smooth  $\infty$ -normed space. Then  $V$  is isometrically order isomorphic onto  $A_0(Q(V))$ .*

*Proof.* Let us define a map  $\phi : V \mapsto A_0(Q(V), V^*)$  given by  $\phi(v) = \tilde{v}$ , where  $\tilde{v}(f) = f(v)$  for all  $f \in V^*$ . It is easy to note that the map  $\phi$  is well defined and linear. Since  $\|v\| = \sup\{|f(v)| : f \in Q(V)\}$  for all  $v \in V$  and since  $v \geq 0$  if and only if  $f(v) \geq 0$  for all  $f$  in  $Q(V)$ , we may also conclude that  $\phi$  is order isometry. This map is onto because any element of  $V$  can realised as a weak-star continuous linear map on  $V^*$ . Since  $V$  is complete by isometry argument  $A_0(Q(V), V^*)$  is also complete. We prove that  $A_0(Q(V), V^*) = A_0(Q(V))$ .

To see this, let  $a \in A_0(Q(V))$  and  $\epsilon > 0$ . For  $v \in A_0(Q(V))$  and  $\alpha \in \mathbb{R}$ , we define  $G_{Q(V)}(v + \alpha) = \{(f, f(v) + \alpha) : f \in Q(V)\}$ . Then  $G_{Q(V)}(a)$  and  $G_{Q(V)}(a + \epsilon)$  are compact convex sets in  $V' \times \mathbb{R}$ . Also,  $G_{Q(V)}(a) \cap G_{Q(V)}(a + \epsilon) = \emptyset$ . Thus by Hahn Banach Separation theorem there is continuous linear functional  $(f, \lambda)$  on  $V' \times \mathbb{R}$  such that

$$f(x) + \lambda a(x) < f(y) + \lambda a(y) + \lambda \epsilon \quad \forall x, y \in Q(V).$$

Let  $f_0 = -\frac{1}{\lambda}f$ . Then  $f_0 \in A_0(Q(V), V^*)$  such that  $f_0(y) - a(y) < \epsilon$  for all  $y$  in  $Q(V)$  and  $a(x) - f_0(x) < \epsilon$  for all  $x \in Q(V)$ . So we have  $|a(x) - f_0(x)| < \epsilon$  for all  $x \in Q(V)$ . Therefore,  $A_0(Q(V)) = \overline{A_0(Q(V), V^*)}$ , but  $A_0(Q(V), V^*)$  is complete, so  $A_0(Q(V)) = A_0(Q(V), V^*)$ .  $\square$

We prove that the two order unit spaces  $\tilde{V}$  and  $A(Q(V))$  are isometrically order isomorphic to each other. Thus  $A(Q(V))$  may be realised as the concrete order unit space obtained by adjoining an order unit to  $V$ . The following result has a standard proof which we shall omit.

**Lemma 4.8.** *If  $K$  and  $S$  be two compact convex sets and  $\phi : K \mapsto S$  be a affine homeomorphism. Then the map  $\Psi : A(S) \mapsto A(K)$ , defined by  $\psi(a) = a \circ \phi$  for all  $a \in A(S)$  is a isometrical order isomorphism.*

**Theorem 4.9.** *Let  $V$  be a complete order smooth  $\infty$ -normed space. Then  $A(Q(V))$  is isometrically order isomorphic to  $\tilde{V}$ , the order unit space obtained by adjoining an order unit to  $V$ .*

*Proof.* Let  $V$  be a complete order smooth  $\infty$ -normed space. We know from theorem 4.7 that  $V$  is isometrically order isomorphic to  $A_0(Q(V))$ . Since  $V$  is complete,  $\tilde{V}$  is also complete. Therefore,  $\tilde{V}$  is isometrically order isomorphic to  $A(S(\tilde{V}))$ , where  $S(\tilde{V})$  is the state space of  $\tilde{V}$ . We show that  $S(\tilde{V})$  is affinely homeomorphic to  $Q(V)$ . To see this, let  $g \in Q(V)$ . Define  $\tilde{g}(v, \alpha) = g(v) + \alpha$  for  $(v, \alpha) \in \tilde{V}$ . If  $(v, \alpha) \in \tilde{V}^+$ , then for  $\epsilon > 0$ , there exist  $u \in V^+$  such that  $u + v \geq 0$  and  $\|u\| < \alpha + \epsilon$ . Thus

$$\tilde{g}(v, \alpha) \geq g(v) + \alpha \geq -g(u) + \alpha \geq -\|u\| + \alpha > -\epsilon.$$

Since  $\epsilon$  is independent of  $g$ , we see that  $\tilde{g}$  is positive linear map on  $\tilde{V}$  with  $\tilde{g}(0, 1) = 1$ . So  $\tilde{g} \in S(\tilde{V})$ . Further, if  $h \in S(\tilde{V})$  is any extension of  $g$ , then  $h(0, 1) = 1 = \tilde{g}(0, 1)$  so that  $\tilde{g} = h$ . Thus each  $g \in Q(V)$  has a unique extension  $\tilde{g} \in S(\tilde{V})$  and consequently, we obtain a well defined and bijective map  $\phi : Q(V) \mapsto S(\tilde{V})$  by  $\phi(f) = \tilde{f}$ , where  $\tilde{f}(v, \alpha) = f(v) + \alpha$ . Now, it is routine to check that  $\phi$  is affine as well as  $w^*$ - $w^*$  homeomorphism. So by Lemma 4.8,  $A(Q(V))$  is isometrically order isomorphic to  $A(S(\tilde{V}))$ .  $\square$

We know from theorem 4.5 that every approximate order unit space  $(V, \{e_\lambda\})$  is an  $M$ -ideal in the order unit space  $(\tilde{V}, e)$  obtained by adjoining an order unit to  $V$ . In the next theorem, we prove that it is a necessary condition as well.

**Theorem 4.10.** *Let  $V$  be complete order smooth  $\infty$ -normed space. Then  $V$  is an  $M$ -ideal in  $\tilde{V}$  iff  $V$  is an approximate order unit space.*

*Proof.* Let  $V$  be a complete order smooth  $\infty$ -normed space. By Theorems 4.7 and 4.9,  $V$  is an  $M$ -ideal in  $\tilde{V}$  iff  $A_0(Q(V))$  is an  $M$ -ideal in  $A(Q(V))$ . If  $V$  is an approximate order unit space, then by Corollary 4.5,  $V$  is an  $M$ -ideal in  $\tilde{V}$ . Conversely, assume that  $A_0(Q(V))$  is an  $M$ -ideal in  $A(Q(V))$ . We shall prove that  $A_0(Q(V))$  is an approximate order unit space. First, note that  $A_0(Q(V))^\perp \cap Q(V) = \{0\}$ . In fact, if  $f \neq 0$  in  $Q(V)$ , then by the Hahn Banach Separation theorem, there exists  $a \in A_0(Q(V))$  such that  $a(f) \neq 0$ .

Next, we show that  $\{0\}^C = S(V)$  so that  $S(V)$  would be a (split) face of  $Q(V)$ . By the definition of the complementary set, we have

$$\{0\}^C = \cup \{F \subset Q(V) : F \text{ is a face of } Q(V), 0 \notin F\}.$$

If  $F$  is a face of  $Q(V)$  and  $0 \notin F$ , then we have  $F \subseteq S(V)$ . For, if  $f \in F$  and  $\|f\| < 1$ , then we can write  $f = (1 - \|f\|)0 + \|f\|\frac{f}{\|f\|}$ . Thus  $\{0\}^C \subset S(V)$ . Conversely, let  $g \in S(V)$ . If  $h \in \text{face}_{Q(V)}(g)$ . Then by definition of  $\text{face}_{Q(V)}(g)$ , we get  $g = \lambda h + (1 - \lambda)k$ , for some  $k \in Q(V)$  and  $0 < \lambda < 1$ . Now by the triangle inequality,  $\|h\| = \|k\| = 1$  so that  $0 \notin \text{face}_{Q(V)}(g)$  and consequently,  $g \in \text{face}_{Q(V)}(g) \subseteq \{0\}^C$ . Thus  $\{0\}^C = S(V)$ .

Finally, let  $f, g \in V'^+$ . Since  $V$  is an  $M$ -ideal,  $S(V) (= \{0\}^C)$  is convex. Thus

$$1 = \left\| \frac{\|f\|}{\|f\| + \|g\|} \left( \frac{f}{\|f\|} \right) + \frac{\|g\|}{\|f\| + \|g\|} \left( \frac{g}{\|g\|} \right) \right\| = \frac{\|f + g\|}{\|f\| + \|g\|}.$$

Therefore, norm is additive in  $V^{++}$  so that  $V^*$  is a base normed space. In other words,  $V$  is an approximate order unit space.  $\square$

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